STOCHASTIC PROCESSES IN ECONOMIC MODELS OF ASSET BUBBLES

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1. Introduction

Asset bubbles and their subsequent crashes have puzzled historians, economists, financiers and the general populous throughout history. Examples, often categorized as bubbles, include Tulip Mania, the South Seas bubble, the Dot Com bubble, and the recent housing bubble. The importance of bubbles and crashes cannot be overlooked, as demonstrated by the recent housing bubble: as a consequence of the crash, global GDP (the cumulative GDP of every country) was severely affected. Much of the literature in macroeconomics had ignored the consequences of bubbles by writing off the financial sector completely and instead focused on modeling industrial and technology sectors (termed the real economy). Although they acknowledged the role of the financial sector in providing funds to the firms that comprised the real economy, they essentially assumed away any problems arising from financial bubbles. In light of the recent crises, the literature is now shifting toward an approach that brings together financial economics, monetary economics, and standard macroeconomic techniques.

To study asset bubbles, we must first have some notion of how to formally define them. An asset bubble is formed when an asset's price is significantly different from its fundamental value, or intrinsic value as it is also known. The fundamental or intrinsic value can be defined in two equivalent ways. The first is the expected (under a risk neutral measure) present value of its future cash flows (dividends and maturity value, if any). The second is the price, after purchase, you would pay if you had to hold the asset for its entire lifetime, or until maturity. If you reflect on these, it becomes clear that they are equivalent definitions. A difference between the market price and the fundamental value only takes place if one *expects* that selling (re-trading) can generate a higher value than holding it until maturity (possibly infinite) and this difference is the bubble. The terminology reflects the popular belief that such deviations are short lived and at some point surely burst, like a soap bubble. It has been shown that many internet stocks exhibited a price bubble during the nineties when they soared to astonishingly high values despite continued reports of zero or negative earnings and in spite of the fact that there were no dividend payments.

There have been many insightful theories put forth as to the economic causes of bubbles. Economist Hyman Minsky provided an early, informal characterization of bubbles and the associated bursts. In his characterization, Minsky distinguishes between five phases (see, for example, the description of Minsky's model in Kindleberger [7]). An initial displacement for example, a new technology or financial innovation, leads to expectations of increased profits and economic growth. This leads to a boom phase that is usually characterized by low volatility, credit expansion, and increases in investment. Asset prices rise, slowly in the beginning but then with growing momentum. These increases in prices may be such that prices start exceeding the actual fundamental improvements from the innovation. A phase of euphoria then follows, during which investors trade the overvalued asset in a frenzy. Prices increase in an explosive fashion, at which point investors may be aware, or at least suspicious, that a bubble exists, but they remain confident that they can sell the asset to a greater fool in the future. Usually, this phase will be associated with high trading volume. The resulting trading frenzy may also lead to price volatility as observed, for example, during the internet bubble of the late 1990s. At some point, sophisticated investors start cutting back their positions and taking their profits. During this phase of profit taking there may, for a while, be enough demand from less sophisticated investors who may be new to that particular market. However, at some point prices start to fall rapidly, leading to a panic phase, in which investors dump the asset. Prices spiral down, often accelerated by margin calls and weakening balance sheets–the bubble bursts. The more modern literature in economics has formally modeled bubbles arising from differences in beliefs of investors and short-sales constraints; and herding behavior where an initial price increase is viewed as a good signal, more people buy, driving the price up a bit more, which is interpreted as another good signal, and so on. Another interesting idea that has been formally modeled is that of sequential awareness and "riding the bubble." This concept attempts to explain why, once a bubble exists, it may persist for some time. The idea is that investors aren't initially aware of the bubble, but sequentially find out. However, even after an investor becomes aware of the bubble, she may choose to hold her position in the asset rather than selling–riding the bubble–depending on her beliefs about how many other investors know of the bubble. This process can allow the bubble to persist for some time. For the interested reader, good surveys on the modern bubbles literature in economics include Brunnermeier [1][2], Camerer [3], and Kindleberger [7].

We will not be addressing this literature, but will instead take a more "reduced form" approach, where we do not explicitly model the underlying economic phenomena, but instead analyze prices to try to determine whether or not a bubble has formed. In particular, we analyze what properties of general price processes admit and imply bubbles under a no arbitrage assumption which we impose below.

The organization of this paper is as follows: In Section 2, we discuss the assumptions, basic model components and basic definitions such as no arbitrage, risk neutral measure, and market completeness. In addition, we also present some results under a simplified setting such as the first and second fundamental theorems of asset pricing In Section 3, we briefly discuss a stronger version of no arbitrage termed No Free Lunch with Vanishing Risk (NFLVR), characterize bubbles using strict local martingales, provide a theorem classifying bubbles into three types, and discuss how bubbles cannot be born in a complete market, that is they either exist at the beginning of time, or not at all. In Section 4 we discuss incomplete markets and a method of dynamically choosing, at stopping times, different risk neutral measures. In Section 5, we continue in the incomplete market setting and detail how this framework allows for the birth of a bubble. Finally, in Section 6, we briefly recap and conclude.

2. Assumptions, Basic Definitions, and the Fundamental Theorems of Asset Pricing

In this section, we use Jarrow and Chatterjea's book [6] as a reference for the economic assumptions underlying our model and Shreve's book [9] as a reference for an exposition of the first and second fundamental theorems of asset pricing. Naturally, we first put forth the assumptions:

A1. No market frictions. We make the assumption that trading does not involve market frictions such as transactions costs (brokerage commissions, bid-ask spreads, and the price impact of a trade), margin requirements, short sales restrictions, fees charged by exchanges, and taxes (which may be levied on different securities at different rates). In addition assets are perfectly divisible. This allows us to create a benchmark model wherein one can later add frictions, and it is a reasonable approximation for institutional traders.

A2. No credit risk. We assume that no credit risk exists. Also known as default risk or counterparty risk, this is the risk wherein the counterparty (the party on the other side of the derivative contract) will fail to perform on an obligation. The absence of credit risk is a reasonable assumption for exchange-traded derivatives. These are standardized derivative contracts which are traded on an exchange and have well established collateral provisions specified so as to mitigate default risk. Examples of exchange traded derivatives include European call and put options. Over-the-counter (OTC) derivatives are private derivative contracts between two parties that are tailored to the hedging (or speculative) needs of the parties involved. OTC derivatives are not nearly as regulated as exchange traded derivatives and they may possess substantial credit risks unless there are adequate collateral provisions. Examples of OTC derivatives include the Mortgage Backed Securities (MBSs) and Credit Default Swaps (CDSs) often attributed to facilitating the recent collapse of the financial system.

A3. Competitive markets and equivalent beliefs. In a competitive market, traders' purchases and sales don't have any impact on the market price, accordingly, traders act as price takers. Equivalent beliefs signify that traders agree upon zero probability events. The competitive markets assumption is the dynamo of modern economics. When traders disagree on such zero probability events, they cannot come to a consensus regarding what is considered an arbitrage opportunity. This destroys our key tool for proving results (see assumption A5 below).

A4. No intermediate cash flows. Many assets compensate their holders with income or cash flows such as dividends for stocks and coupon income for bonds. The Black-Scholes-Merton model assumes that the underlying asset has no cash flows over the options life. This will be relaxed next section.

A5. No arbitrage opportunities An Arbitrage Oppotunity is defined as a chance to make riskless profits with no investment. Depending on the processes we are dealing with, there are a couple ways to formalize this assumption: the classical no arbitrage and the NFLVR formulations defined below.

A6. Complete Filtered Probability Space We assume our initial (objective) filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{F}, P)$ satisfies the usual conditions. In addition, we assume that the filtration is the completion of the filtration generated by d-dimensional Brownian motion.

A7. Stock price follows generalized geometric Brownian motion (GGBM). In this section, we assume there are m stocks (S_i), each represented by the SDE

$$dS_i(t) = \alpha_i(t)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)W_j(t), \quad i = 1, ..., m \tag{1}$$

where $\alpha_i(t)$ and $\sigma_{ij}(t)$ are adapted processes. This assumption is often relaxed in the literature to include much more general semimartingale processes with jumps. In the following sections, we will be dealing with cadlag $L^1(\mathbb{Q})$ ($\mathbb{Q}$ is the risk neutral probability measure defined below) semimartigale asset prices. In that case, the results in this section follow with some modification (see below).

A8. Interest rate process and money market account (MMA). We denote the instantaneous default-free spot interest rate by $r = (r(t))_{t\geq 0}$ and we assume it to be predictable. The time t value of a money market account (MMA) is defined to be

$$B(t) = \exp\left(\int_0^t r(u)du\right), \tag{2}$$

with initial value $B(0) = 1$. We define the discount process $D(t) := B(t)^{-1}$.

Next, we define the notion of a Risk-neutral measure.

Definition 2.1 (Risk-Neutral Measure)**.** *A probability measure $\mathbb{Q}$ is said to be risk-neutral if*

(i) *$\mathbb{Q}$ and $\mathbb{P}$ are equivalent (i.e. for every $A \in \mathcal{F}$ $\mathbb{P}(A) = 0$ if and only if $\mathbb{Q}(A) = 0$)*

(ii) *Under $\mathbb{Q}$, the discounted stock price $D(t)S_i(t)$ is a (local) martingale for every $i = 1, ..., m$.*

We would like to note that "(local)" was added to accommodate the more general setting which applies in the sections below, however, in this section we take the more restrictive definition.

The dynamics of a discounted stock price process DS_i are

$$\begin{aligned} d(D(t)S_i(t)) &= D(t)\left(dS_i(t) - r(t)S_i(t)dt\right) \\ &= D(t)S_i(t)\left[(\alpha_i(t) - r(t))dt + \sum_{j=1}^{d}\sigma_{ij}dW_j(t)\right] \end{aligned} \tag{3}$$

In order to make the discounted stock prices martingales, we would like to rewrite (3) as

$$d(D(t)S_i(t)) = D(t)S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)[\theta_j(t)dt + dW_j(t)] \tag{4}$$

where θ_j are called the market price of risk processes. The name can most readily be interpreted when $d = 1$, in which case $\theta(t) = (\alpha(t) - r(t))/\sigma(t)$, which is interpreted to be the premium of investing in the stock compared to a bond (mean rate of return of the stock - the interest rate) per unit of risk. If we can find the market price of risk processes that make (4) hold, we can then use the multi-dimenstional Girsanov Theorem (see Theorem 5.4.1 in

[9]) to construct an equivalent probability measure $\mathbb{Q}$ under which

$$W^*(t) = W(t) + \int_0^t \Theta(u)du \tag{5}$$

where $\Theta(t) = (\theta_1(t), ..., \theta_d(t))$, is a d-dimensional Brownian motion. Thus, we can express (4) as

$$d(D(t)S_i(t)) = D(t)S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t)dW_j^*(t) \tag{6}$$

and hence it is a martingale under $\mathbb{Q}$. Thus, we've reduced our problem to that of finding processes θ_j that make (3) and (4) agree, which occurs if and only if

$$\alpha_i(t) - r(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\theta_j(t), \quad i = 1, ..., m \tag{7}$$

So we have m equations in d unknown processes. We call these equations the market price of risk equations.

Next, we introduce the a basic building block of the theory, the portfolio value process.

Definition 2.2 (Portfolio Value Process). *A portfolio value process X is defined by the SDE*

$$\begin{aligned} dX(t) &= \sum_{i=1}^{m} \Delta_i(t)dS_i(t) + \eta_t dB(t) \\ &= \sum_{i=1}^{m} \Delta_i(t)dS_i(t) + r(t)\left(X(t) - \sum_{i=1}^{m} \Delta_i(t)S_i(t)\right)dt \end{aligned} \tag{8}$$

with initial value $X(0) = X_0$ (a constant) and where Δ_i represents the share holdings in asset i and the coefficient of dt represents the whatever remains after investing in the stock and assumed to be saved in the MMA.

We also have to rule out the classic doubling strategy in gambling. Here a player bets an initial amount and keeps on doubling this bet if he loses and stops when he wins. This strategy leads to a riskless gain. However, for the agent to be able to make this riskless profit, he would need to be able to tolerate arbitrarily large losses. To rule this out, we make the following assumption of admissibility:

Definition 2.3 (α-admissible). *Let $\alpha > 0$, and let S be a semimartingale. A predictable trading strategy Δ is α-admissible if $\Delta_0 = 0$ and $\int_0^t \Delta_s dS(s) \geq -\alpha$ for all $t \geq 0$. Δ is called admissible if there exists α s.t. Δ is α-admissible.*

Next, we will show an interesting result, that under the risk neutral measure $\mathbb{Q}$, the discounted portfolio value process $D(t)X(t)$ is a martingale. That is, the measure with the defining property that our discounted stock price process is a martingale, also makes any discounted portfolio value process a martingale.

Lemma 2.1. *Let $\mathbb{Q}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\mathbb{Q}$, the discounted portfolio value $D(t)X(t)$ is a (local) martingale.*

Proof. The SDE above (8) can be expressed as:

$$\begin{aligned} dX(t) &= r(t)X(t)dt + \sum_{i=1}^{m} \Delta_i(t)\left[dS_i(t) - r(t)S_i(t)\right]dt \\ &= r(t)X(t)dt + \sum_{i=1}^{m} B(t)d(D(t)dS_i(t)) \end{aligned} \tag{9}$$

The differential of the discounted portfolio value is

$$d(D(t)X(t)) = D(t)(dX(t) - r(t)S_i(t)dt) = \sum_{i=1}^{m} \Delta_i(t)d(D(t)S_i(t)) \tag{10}$$

□

Thus, if $\mathbb{Q}$ is a risk-neutral measure, then under $\mathbb{Q}$ the processes $D(t)S_i(t)$ are martingales, and hence the process $D(t)X(t)$ must also be a martingale. Put another way, under $\mathbb{Q}$ each of the stocks has mean rate of return $R(t)$, the same as the rate of retum of the money market account. Hence, no matter how an agent invests, the mean rate of return of his portfolio value under $\mathbb{Q}$ must also be $R(t)$, and the discounted portfolio value must then be a martingale. We have proved the following result.

Definition 2.4. *An arbitrage is a portfolio value process $X(t)$ satisfying $X(0) = 0$ and also satisfying for some time $T > 0$*

$$\mathbb{P}\{X(T) \geq 0\} = 1,\ \mathbb{P}\{X(T) > 0\} > 0. \tag{11}$$

An arbitrage is a way of trading so that one starts with zero capital and at some later time T is sure not to have lost money and furthermore has a positive probability of having made money. Such an opportunity exists if and only if there is a way to start with positive capital $X(0)$ and to beat the money market account. In other words, there exists an arbitrage if and only if there is a way to start with $X(0)$ and at a later time T have a portfolio value satisfying

$$\mathbb{P}\left\{X(T) \geq B(T)X(0)\right\} = 1,\ \mathbb{P}\left\{(T) > B(T)X(0)\right\} > 0. \tag{12}$$

Now we are ready to state and prove the first and second fundamental theorems of asset pricing. In what follows, we provide the proof for the first fundamental theorem, but not the second to save on time and space. We would like to note that the proof is quite interesting and heavily relies on the martingale representation theorem and if the reader is interested, Shreve ([9], Theorem 5.4.9) is an excellent reference for it.

Theorem 2.1 (First fundamental theorem of asset pricing)**.** *If a market model has a risk-neutral probability measure, then it does not admit arbitrage (NFLVR–defined below, for general case).*

Proof. Let $\mathbb{Q}$ be a risk-neutral probability measure of our market model. Then every discounted portfolio value process is a martingale under $\mathbb{Q}$. In particular, every portfolio value process has to satisfy:

$$\mathrm{E}_{\mathbb{Q}}[D(T)X(T)] = X(0). \tag{13}$$

Let $X(t)$ be a portfolio value process with $X(0) = 0$. Then we have

$$\mathrm{E}_{\mathbb{Q}}[D(T)X(T)] = 0. \tag{14}$$

Suppose $X(T)$ satisfies the first part of (11) (i.e., $\mathbb{P}\{X(T) < 0\} = 0$). Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we have also $\mathbb{Q}\{X(T) < 0\} = 0$. This, combined with (14), implies $\mathbb{Q}\{X(T) > 0\} = 0$, for otherwise we would have $\mathbb{Q}\{D(T)X(T) > 0\} > 0$, which would imply $\mathrm{E}_{\mathbb{Q}}[D(T)X(T)] > 0$. Because $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, we also have $\mathbb{P}\{X(T) > 0\} = 0$. Hence, $X(t)$ is not an arbitrage. In fact, there cannot exist an arbitrage since every portfolio value process $X(t)$ satisfying $X(0) = 0$ cannot be an arbitrage. □

Definition 2.5 (Market Completeness and Incompleteness). *A market model is said to be complete if every derivative security can be hedged and incomplete otherwise.*

Theorem 2.2 (Second fundamental theorem of asset pricing). *Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique. Furthermore, the model is incomplete if and only if there exist an infinite number of risk-neutral probability measures.*

3. Characterization of Bubbles

We work on a time interval $[0, T^*]$, where $T^* \in [0, \infty]$. It is more interesting to consider a compact time interval (the finite horizon case, where $T^* = T < \infty$), but we'll consider the general case. To simplify things, from now on, we only consider one risky asset. Let τ be the lifetime of the risky asset, where τ is a stopping time, and $\tau \leq T^*$, τ can occur due to bankruptcy, to a buyout of the company by another company, to a merger, to being broken up by antitrust laws, etc.

Let $D = (D_t)_{0 \leq t \leq \tau} \geq 0$ be the cumulative dividend process, and we assume it is a nondecreasing cadlag semimartingale. Denote the ex-cash flow price process of the risky asset by $S = (S_t)_{0 \leq t \leq \tau} \geq 0$, assuming it is a cadlag semimartingale. By ex-cash flow we mean that the price at time t is after all dividend have been paid, including the time t dividend. Let $L \in \mathcal{F}_\tau$ be the time τ terminal payoff or liquidation value of the asset. We assume $L \geq 0$. In addition, as S is ex-cash flow, it must be that after all cash flows have been payed, i.e. $t \geq \tau$, it must be that $S_t = 0$. As a simplifying assumption for the purposes of this exposition, we also assume that all the above variables are also in $L^1(\mathbb{Q})$. The wealth process W associated with the market price of the risky asset plus accumulated cash flows is:

$$W_t = 1_{t \leq \tau} S_t + B_t \int_0^{t \wedge \tau} \frac{1}{B_u} dD_u + \frac{B_t}{B_\tau} L 1_{\tau \leq t} \tag{15}$$

This process corresponds to being endowed with a single share of the risky asset, reinvesting all of its dividends, and finally selling it at time τ. The idea behind considering the wealth process is to see what it would be worth to someone to hold the asset until maturity. As we discussed in the introduction, the "expected" wealth associated with holding the asset until maturity is closely related to intuitive definition of a bubble. Following in suit with the rest of finance, this expectation will of course be with respect to the risk neutral measure $\mathbb{Q}$.

In the previous section our definition of arbitrage was sufficient to prove the first and second *Fundamental Theorem of Asset Pricing* in our simplified setting. However, when considering more general price processes, such as cadlag $L^1(\mathbb{Q})$ semimartingales, we need a stranger notion of no arbitrage namely *No Free Lunch with Vanishing Risk* often referred to by its acronym $NFLVR$. Before defining it, we must first define self-financing trading strategy (SFTS)

Definition 3.1 (Self-Financing Trading Strategy (SFTS)). *A trading strategy* (Δ, η) *is said to be self-financing if*

$$\eta(t)B(t) + \Delta(t)S(t) = \eta_0 B(0) + \Delta(0)S(0) + \int_0^t \eta(s)dB(s) + \int_0^t \Delta(s)dS(s) \tag{16}$$

for all t.

This definition says that the value of your portfolio today (LHS of (16)) is equal to how much you had in your MMA and risky asset to start off with plus the gains or losses from investments in the MMA and risky asset. The fact that we have (16) for all t tells us that we start we an inital amount of capital and only rebalance our portfolio from the proceeds we have at each t.

Now we are ready to define NFLVR:

Definition 3.2 (No Free Lunch with Vanishing Risk (NFLVR)). *A market consisting of the assets* (B, S) *is said to satisfy the assumption of No Free Lunch with Vanishing Risk (NFLVR) if there is no sequence* $f_n = \int_0^{T^*} \eta_s^n dB(s) + \int_0^{T^*} \Delta_s^n dS(s)$, *where the trading strategies* (η^n, Δ^n) *are all admissible and self-financing, s.t.*

$$\|\max(-f_n, 0)\|_\infty \to 0 \quad \textit{and} \quad f_n \to f \tag{17}$$

for some $f \geq 0$ *with* $\mathbb{P}(f > 0) > 0$.

This assumption rules out sequences of trading strategies which converge to arbitrage opportunities. Under this assumption, the theorems hold with local martingales replaced by the martingale assumptions in the respective theorems. Thus, there exists at least one probability measure $\mathbb{Q}$, which is equivalent with $\mathbb{P}$, such that under $\mathbb{Q}$ we have that W is a local martingale. In the case of a complete market the probability measure $\mathbb{Q}$ is unique, whereas in an incomplete market there may be infinitely many probability measures $\mathbb{Q}$. In this section, we will assume that markets are complete and work with a unique risk-neutral probability measure $\mathbb{Q}$.

We define the market's *fundamental value* for the risky asset, which serves as the best guess of the future discounted cash flows.

$$S_t^* = E_{\mathbb{Q}}\left(\int_0^{\tau \wedge T^*} \frac{1}{B_u}\,\mathrm{d}D_u + \frac{L}{B_\tau} 1_{\tau \leq T^*} | \mathcal{F}_t\right) B_t. \tag{18}$$

Definition 3.3 (Bubble Process). *We define* β_t *by*

$$\beta_t = S_t - S_t^*,$$

the difference between the market price and the fundamental price. The process is called a bubble.

In what follows, we take $B = 1$, that is 0 interest rate on the MMA, in order to reduce burdensome notation in proofs. We next show that, in this framework, bubbles must be non-negative. In what follows, we consider only the case where the stock pays no dividends, and the spot interest rate is 0, for simplicity; results still carry though with them being nonzero.

Theorem 3.1. *Let S be the nonnegative price process of a stock and assume S pays no dividends. Moreover assume the spot interest rate is constant and equal to 0. Let $\mathbb{Q}$ be a risk neutral measure under which S is a local martingale (and hence a supermartingale). Let $S^\star$ be the fundamental value of the stock calculated under $\mathbb{Q}$, and let $\beta_t = S_t - S^\star_t$. Then $\beta \geq 0$.*

Proof. Under these simplifying hypotheses of no dividends and 0 interest, the fundamental value of the stock is nothing more than

$$S^\star_t = E_{\mathbb{Q}}(L1_{\{\tau \leq T^\star\}}|\mathcal{F}_t). \tag{19}$$

Since under $\mathbb{Q}$ the process S is a supermartingale, we have

$$E_{\mathbb{Q}}(S_\tau|\mathcal{F}_t) \leq S_t \tag{20}$$

and since $S_\tau = L1_{\{\tau\leq T^\star\}}$, combining (19) and (20) gives the result. ☐

We can classify bubbles into three types, as shown in the following theorem. For this theorem, we assume fixed a risk neutral measure $\mathbb{Q}$ under which both S and W are local martingales.

Theorem 3.2. *If in an asset's price there exists a bubble $\beta = (\beta_t)_{t\geq 0}$ that is not identically zero, then we have three and only three possibilities:*

1. *β_t is a local martingale (which could be a uniformly integrable martingale) if $\mathbb{P}(\tau = \infty) > 0$ ($\mathbb{Q}(\tau = \infty) > 0$).*
2. *β_t is a local martingale but not a uniformly integrable martingale if τ is unbounded, but with $\mathbb{P}(\tau < \infty) = 1$ ($\mathbb{Q}(\tau < \infty) = 1$).*
3. *β_t is a strict $\mathbb{Q}$-local martingale, if τ is a bounded stopping time.*

Proof. Before we begin, we need the following technical lemmas:

Lemma 3.1. *The fundamental price in (4) is well deffined. Furthermore, $(S_t)_{t\geq 0}$ converges to $S_\infty \in L^1(Q)$a.s. and $(S^*_t)_{t\geq 0}$ converges to 0 a.s.*

Lemma 3.2. *The fundamental wealth process $W^* = (W_t)_{t\geq 0}$ given by $W^*_t = S^*_t + \int_0^{\tau\wedge t} dD_u + X_\tau 1_{\{\tau\leq t\}}$ is a uniformly integrable martingale under $\mathbb{Q}$ closed by*

$$W^*_\infty = \int_0^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}.$$

Since W is a nonnegative local martingale under $\mathbb{Q}$, it is also a supermartingale. By the martingale convergence theorem, there exists $W_\infty \in L^1(\mathbb{Q})$ s.t. W converges to W_∞ $\mathbb{Q}$-a.s. Let

$$\beta'_t = W_t - E_Q[W_\infty|\mathcal{F}_t]. \tag{21}$$

Then $(\beta'_t)_{t\geq 0}$ is a (nonnegative) local martingale, because it is a difference of a local martingale and a uniformly integrable martingale. By Lemmas 3.1 and 3.2, we can write

$$E_Q[W_\infty|\mathcal{F}_t] = E_Q[W^*_\infty|\mathcal{F}_t] + E_Q[S_\infty|\mathcal{F}_t] = W^*_t + E_Q[S_\infty|\mathcal{F}_t]. \tag{22}$$

By the definition of wealth processes, and using (21) and (22),

$$
\begin{aligned}
\beta_t &= S_t - S_t^* \\
&= W_t - W_t^* \\
&= (E_{\mathbb{Q}}[W_\infty|\mathcal{F}_t] + \beta_t^1) - (E_{\mathbb{Q}}[W_\infty|\mathcal{F}_t] - E_{\mathbb{Q}}[S_\infty|\mathcal{F}_t]) \\
&= \beta_t' + E_{\mathbb{Q}}[S_\infty|\mathcal{F}_t]. \qquad (23)
\end{aligned}
$$

If $\tau < T$ for $T \in \mathbb{R}^+$, then $S_\infty = 0$. A bubble $\beta_t = \beta_t' = 0$ for $t \geq \tau$ and in particular $\beta_T = 0$. If β_t is a martingale,

$$\beta_t = E_{\mathbb{Q}}[\beta_T|\mathcal{F}_t] = 0 \ \forall t \leq T.$$

It follows that β is a strict local martingale. This proves Part 3. For Part 2, assume that β is a uniformly integrable martingale. Then by Doob s optional sampling theorem, for any stopping time $\tau_0 \leq \tau$,

$$\beta_{\tau_0} = E_{\mathbb{Q}}[\beta_\tau|\mathcal{F}_{\tau_0}] = 0$$

and because β is optional, it follows from the section theorems of P.A. Meyer that $\beta = 0$ on $[0,\ \tau]$. Therefore the bubble does not exist. For Part 1, $(E_Q[S_\infty|\mathcal{F}_t])_{t\geq 0}$ is a uniformly integrable martingale. □

We see a common theme above: if a bubble exists, it must be that the bubble itself is a local martingale. Depending on conditions on our stopping time, we can say more about the necessary condition for a bubble's existence. However, we would like to say something more about bubbles, in particular, we would like to find a sufficient condition for the existence of a bubble. In the case of a uniformly bounded liquidation time τ, we can obtain this without much effort: since $S^\star$ is a true martingale, and $\beta = S - S^\star$, we have that β being a strict local martingale is equivalent to the price process S being a strict local martingale:

Corollary 3.1. *We have a bubble on $[0,T]$ if and only if the price process S is a strict local martingale.*

For this important special case of a bounded horizon (that is, we are working on a compact time interval, $[0,T]$), we can summarize as follows:

Theorem 3.3. *Any non-zero asset price bubble β on $[0,T]$ is a strict $\mathbb{Q}$-local martingale with the following properties:*

1. *$\beta \geq 0$,*
2. *$\beta_\tau = 0$,*
3. *if $\beta_t = 0$ then $\beta_u = 0$ for all $u \geq t$, and*

for any $t \leq T \leq \tau \leq T^\star$.

This theorem states that the asset price bubble β is a strict $\mathbb{Q}$-local martingale. We interpret each of the conditions:

- Condition (1) states that bubbles are always non-negative, i.e. the market price can never be less than the fundamental value.
- Condition (2) states that the bubble must burst on or before τ.
- Condition (3) states that if the bubble ever bursts before the asset's maturity, then it can never start again. Thus, in complete markets, since we just have one risk-neutral measure, a bubble has to either exist at the beginning of our modeling period, or not at all.

4. Choosing a Risk-Neutral Measure in Incomplete Markets

From Section II, the Second Fundamental Theorem of Asset Pricing gives us that a market is incomplete if and only if there exists an infinite number of equivalent risk neutral measures. The question then naturally arises: How do we choose a risk-neutral measure so that we can well define the fundamental value of a risky asset? Several approaches have been proposed, such as indifference pricing or choosing a risk neutral measure by choosing one that minimizes the entropy (or alternatively the "distance") between the objective measure and the class of risk neutral measures, by minimizing the variance of certain terms in the semimartingale decomposition, known as choosing the minimal variance measure (see for example Follmer and Schweizer [4]). The one used in the bubbles literature surveyed here is slightly more intuitive: take an incomplete market, add options, so that we can hedge other assets, until we complete it, giving rise to a unique risk-neutral measure, which we then use in the original incomplete market for valuation (see Jacod and Protter [5]).

In this section, we begin by assuming the market consisting of (B, S) is incomplete, thus we have an infinite choice of risk neutral measures, and we call this set $\mathbb{Q}_S$, where the subscript S reminds us that not all derivatives with underlier S can be hedged, giving rise to the incompleteness of the market. We then think about the situation in which sufficiently many options are added to complete the market. Once these options are in place, given prices of the options, we know that there will exist a unique risk neutral measure. However, once prices of these added options change, for whatever reason, so can the unique risk neutral measure. As we will see next section, it is this flexibility that allows Protter and Jarrow among others to model bubble birth in an incomplete market.

In this section, we will restrict our attention to a single stock. We assume the following dynamics for the stock price:

$$S_t = S_0 + \int_0^t \alpha_s ds + \sum_{i \in I} \int_0^t \sigma_s^i dW_s^i \tag{24}$$

where a and σ^i are assumed to be predictable and satisfy

$$\int_0^t (|\alpha_s| + \sum_{i \in I} |\sigma_s^i|^2) < \infty \ a.s. \tag{25}$$

for all t. In addition, we assume that the coefficients are Lipschitz continuous functions of S_t s.t the solution $S_t > 0$. One such example would be the GGBM considered in Section 2.

For the added options, we consider a fixed payoff function g on $(0, \infty)$ which is nonnegative and convex, and we denote by $P(T)_t$ the price at time $t \in [0, T]$ of the option with pay-off $g(S_T)$ at maturity date T. We also assume that g is not affine, otherwise $P(T)_t = g(S_t)$ and we are in a trivial situation.

The idea of this section is to complete the market by adding enough European calls. We can do this since any option with convex payoffs can be expressed as:

$$P(T)_T = g(0) + {g'}_+(0) S_T + \int_0^\infty (S_T - K)^+ \mu(dK), \tag{26}$$

where μ is a positive measure on $\mathbb{R}$ with $\mu = g''$, where the mathematical derivative is in the generalized function sense.

We denote by $\mathcal{T}$ the set of maturity dates T corresponding to the tradable options (always with the same given pay-off function g), and by $T_\star$ the time horizon up to when trading may take place. Even when $T_\star < \infty$, there might be options with maturity date $T > T_\star$, so we need to specify the model up to infinity.

In practice $\mathcal{T}$ is a finite set, although perhaps quite large. For the mathematical analysis it is much more convenient to take $\mathcal{T}$ to be an interval, or perhaps a countable set which is dense in an interval. We consider the case where $T_\star < \infty$ and $\mathcal{T} = [T_0, \infty)$, with $T_0 > T_\star$.

Next, we want to express $P(T)_T = g(S_T)$, in terms of the same W^i and μ, rather than with S. To do so, we write

$$P(T)_t = P(T_0)_t + \int_{T_0}^{T} f(t,s)ds \tag{27}$$

where f can be thought of as the risk neutral time t conditional expectation of the european call's payoff $(S_T - K)$. Thus, f is an Ito process and can be expressed as

$$f(t,s) = f(0,s) + \int_0^t \alpha(u,s)du + \sum_{i\in I} \int_0^t \gamma^i(u,s)dW_u^i \tag{28}$$

so

$$P(T_0)_t = P(T_0)_0 + \int_0^t \overline{\alpha}_s ds + \sum_{i\in I} \int_0^t \overline{\gamma}_s^i dW_s^i \text{for} \leq T^*. \tag{29}$$

In addition, we assume that the coefficients satisfy the same conditions as above.

An example of the type of results obtained in Protter and Jacod is when trading takes place up to time $T_\star$, and the maturity dates of the options are all $T \geq T_0$, where $T_0 > T_\star$. We denote $\mathcal{M}_{loc}(T_\star, T_0)$ the collection of risk neutral measures for S that are compatible with the option structure so that no arbitrage opportunities exist.

Theorem 4.1. *Consider the fair price model as outlined above such that the set $\mathcal{M}_{loc}(T_\star,\ T_0)$ is not empty. Then this set is a singleton if and only if we have the following property: the system of linear equations*

- $\sum_{i\in I} \sigma_s^i(\omega)\beta_i \ = 0$
- $\sum_{i\in I} \overline{\gamma}_s^i(\omega)\beta_i \ = 0$
- $T \geq T_0 \ \Rightarrow \ \sum_{i\in I}(\alpha^i(s,T)(\omega)\beta_i \ = 0$

where β_i and y are square integrable with respect to t, then $\beta_i = 0$ and $y = 0$ will be its only solution up to a Lebesgue-null set

5. Bubble Birth in Incomplete Markets

In this section, we finally detail how a bubble can be born in our incomplete market model. Throughout this section we assume a fixed finite horizon $[0, T]$. We wish to allow for the switching of risk-neutral measures at totally inaccessible stopping times. These shifting local martingale measures may be interpreted as regime shifts in the underlying economic fundamentals (beliefs, risk-aversion, endowments, technology, institutional structures) or extrinsic factors (animal spirits). The idea of bubble formation in this setting is as follows: we start off with a regime selected by the market as discussed above in which no bubble is admitted, that is, as discussed in Section 3 (finite horizon case) the stock price is not a strict local martingale, but then we randomly switch to a new regime in which the stock becomes a strict local martingale, thus creating a bubble.

We let $(\sigma_i)_{i\geq 0}$ denote an increasing sequence of random times with $\sigma_0 = 0$ representing the times of regime shifts in the economy. We also impose σ_i to be totally inaccessible stopping times, for if they were to be predictable, traders would have the opportunity to develop arbitrage strategies around the shifts. It's interesting to note that in the minimal Brownian paradigm, then there are no totally inaccessible stopping times, so we would need to consider a larger space that supports such times.

We let $(Y^i)_{i\geq 0}$ be a sequence of random variables characterizing the state of the economy at those times (the particular regime's characteristics) such that $(Y^i)_{i\geq 0}$ and $(\sigma)_{i\geq 0}$ are independent of each other. Furthermore, we assume that both $(Y^i)_{i\geq 0}$ and $(\sigma)_{i\geq 0}$ are independent of the underlying filtration $\mathbb{F}$ to which the price process S is adapted.

Define two stochastic processes $(N_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ by

$$N_t = \sum_{i\geq 0} 1_{\{t\geq \sigma_i\}} \textit{ and } Y_t = \sum_{i\geq 0} Y^i 1_{\{\sigma_i \leq t < \sigma_{i+1}\}} \tag{30}$$

N_t counts the number of regime shifts up to and including time t, while Y_t identifies the characteristics of the regime at time t. Let $\mathbb{H}$ be a natural filtration generated by N and Y and define the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ By the definition of $\mathbb{G}$, $(\sigma_i)_{i\geq 0}$ is an increasing sequence of $\mathbb{G}$ stopping times.

Since N and Y are independent of $\mathbb{F}$, every $(Q, \mathbb{F})$-local martingale is also a $(Q, \mathbb{G})-$ local martingale. Thus changing the distribution of N and/or Y will maintain the martingale property of risky asset and, by implication, the wealth process W. Recall $\mathbb{Q}_S$ is the set of all risk neutral measures $\mathbb{Q}$ under which S is a $\mathbb{Q}$-local martingale. and denote $\mathbb{Q}^i \in \mathbb{Q}_s$ as the risk neutral measure "selected by the market" at time t given Y^i. The selection mechanism for a special case was presented in the last section as well as a system of equations that characterized it. However, in that special case, the only options that needed to be added were those that had maturities that lasted beyond the end period of trading. For the general treatment we refer the reader to Jacod and Protter [5], however, we will take it for granted that the market can select a unique measure.

The fundamental price of an asset is defined analogously to the complete market case where we have only one risk-neutral measure, however in our setting, we must account for the shifting regimes:

Definition 5.1 (Fundamental Price). *Let $\phi \in \Phi$ be an asset with maturity ν and payoff (D_ϕ, L^ν_ϕ) (dividends and liquidation value of the asset). The fundamental price $\Lambda^\star_t(\phi)$ of asset ϕ is deffined by*

$$\Lambda^\star_t(\phi) = \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\nu dD_\phi(u) + L^\nu_\phi 1_{\{\tau<\infty\}} \middle| \mathcal{F}_t\right] 1_{\{t<\nu\}\cap\{t\in[\sigma_i,\sigma_{i+1})\}} \tag{31}$$

$\forall t \in [0,\ \infty)$ *where* $\Lambda^\star_\infty(\phi) = 0$.

Thus, the fundamental price of our stock would be given by (recall $B = 1$)

$$S^\star_t = \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\nu dD_u + \Xi^\nu 1_{\{\nu<\infty\}} \middle| \mathcal{F}_t\right] 1_{\{t<\nu\}\cap\{t\in[\sigma_i,\sigma_{i+1})\}} \tag{32}$$

At any time $t < \tau$, given that we are in the i^{th} regime $\{\sigma_i \leq t < \sigma_{i+1}\}$, the right side of (32) simplifies to:

$$S_t^\star = E_{Q^i}\left[\int_t^\tau dD_u + X_\tau 1_{\{\tau<\infty\}}|\mathcal{F}_t\right]$$

Thus, within a regime, before maturity, we see that the fundamental price equals its expected future cash flows, just as before, under our market selected risk-neutral measure. We also note that within a regime, our characterizing results of finite horizon bubbles Corollary 3.1 and Theorem 3.3, still hold.

6. Conclusion

We've set forth a method of modelling bubbles under a fairly general setting developed by Jarrow and Protter among others. In this framework, bubbles are closely linked to strict local martingales. We've also seen that in a complete market bubbles must either exist at the beginning of time or not at all. However, moving to an incomplete market setting we can model the birth of a market by shifting regimes at totally inaccessible stopping times. Then, in the finite horizon case, a birth can occur if we start within a regime for which our risky asset is not a strict martingale and then shift to another regime in which it is.

One may ask: how would one go about detecting such regime changes in the real world? It turns out that using this framework, one can derive certain conditions on traded options prices from which one can detect asset bubbles. Due to time constraints, we were not able to cover that method, however, we encourage the reader, if interested, to refer to Section 10 of [8].

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